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Non-linearity, relaxation and diffusion in acoustics and ultrasonics

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As they propagate through a gas, fluctuating pressure signals of moderate amplitude and of ultrasonic frequency are affected by amplitude dispersion, by relaxation damping and, particularly in 'shock layers', by diffusive damping. We derive a 'high frequency' theory including all these effects, for disturbances of arbitrary wave form excited by a wide variety of boundary conditions. By introducing a phase variable α, and taking account of non-linearity, we show how the signal propagates along the rays of linear acoustics theory, with constantly changing wave profile.

Relaxation dampens the signal, as for linear acoustics, and also diminishes amplitude dispersion. A criterion for shock formation is given, and the importance of non-linearity for signal attenuation exhibited. As shocks form, α surfaces coalesce and diffusive mechanisms are accentuated. Whitham's area rule is shown to be relevant for unsteady three-dimensional flows in relaxing gases, and is used to compute the attentuation of an ultrasonic beam. Supersonic relaxing flow over a wavy wall is also analyzed, and focusing effects are discussed.

1. Introduction

In a relaxing gas acoustic disturbances are affected by three classes of mechanism: (i) those, like vibrational or chemical non-equilibrium, which are 'slow' rate processes having 'relaxation times' much greater than one signal 'period'; (ii) those associated with 'fast' processes, like adjustment to equilibrium distributions of translational and rotational energy modes, for which the relaxation times are much shorter than one signal period; (iii) non-linearity, causing profile distortion, and tending to steepen compressive portions of the signal, until balanced within 'shock layers' by accentuated class (ii) mechanisms. Here, we extend previous investigations (Parker 1968) of class (i) and (iii) effects in one-dimensional signals, to describe the three-dimensional propagation of moderate amplitude oscillatory signals in moving, relaxing gases.

We consider ultrasonic signals having frequencies high compared to the 'relaxation frequency' of some vibrational process of the gas, yet sufficiently low for diffusive effects to be important only in shock layers. Moreover, the translational and rotational energy modes are everywhere taken as sufficiently close to equilibrium for standard viscosity, heat conduction and diffusion processes to describe momentum, energy and composition transport. In such signals the pressure, density and velocity fluctuate rapidly but, outside shock layers, are approximately related as in a simple wave, whilst the vibrational excitation, and also the specific entropy, change significantly only over larger intervals. A 'two time scale' analysis is naturally suggested.

In §3 an auxiliary variable α is introduced, generalizing both the phase variable of linear acoustics and the variable taken to label wavelets of a simple wave. All dependent variables are then defined at *all* $(\alpha, \mathbf{x}, t)$, even though they have physical meaning only for values of t related implicitly to α at each $\mathbf{x}$. The normal $\mathbf{n}(\alpha, \mathbf{x}, t)$ and propagation speed $c(\alpha, \mathbf{x}, t)$ of the physically meaningful surfaces of constant α must then be determined as part of the high frequency procedure. We treat the rapid fluctuations in pressure, density and velocity as small, but do not neglect the associated fluctuations in characteristic speed, so that the small fluctuations in c and $\mathbf{n}$ allow shock formation and decay, but on length scales much greater than a 'wavelength'. Through a high frequency analysis the propagation rays are determined, as functions of $(\mathbf{x}, t)$, and are those of linear acoustics, and along each a limited α interval of the wave form is, to this approximation, appropriately represented. The associated 'propagation equation', including small diffusive terms and governing the signal strength along *each* ray at *each* constant α, generalizes Lighthill's (1956) treatment of one-dimensional gas disturbances satisfying Burgers' equation.

The modulation, distortion and attenuation of the signal is analyzed in §5. Outside shocks, the 'ray' equation for signal strength at each wavelet is that of Bretherton & Garrett (1968) for conservative systems, modified by a relaxation damping term. The effects of background inhomogeneities in the gas, and of refraction and geometric focusing, are thus simply related to those for linearized wave trains in a non-relaxing gas. In this description the amplitude dispersion (or profile distortion) is governed by a second 'ray' equation, whose solutions determine the physically meaningful $(\alpha, \mathbf{x}, t)$. This equation exhibits the competing effects of relaxation, refraction and focusing on the distortion rate, and gives a criterion (similar to Varley & Rogers (1967) for viscoelastic materials) for eventual shock formation. Moreover, once shocks do form their positions within the wave profile may be found directly from Whitham's (1952) area rule. To this approximation, none of the modulation effects can alter the succession of profile shapes associated with any ray. They can effect only the propagation time until certain α wavelets are absorbed in the shock, and the amplitude and wavelength at that point. In 'strong' (or extremely rapid) signals distortion is rapid and profiles soon resemble non-relaxing 'N waves', whilst for weaker waves relaxation damping retards the wave form distortion, and may completely prevent shock formation. These main results are expressed in equations (44), (45), (47) and (52).

In §6 we examine the attenuation of a uni-directional beam passing through a uniform region, showing the importance of class (ii) mechanisms once shocks have formed, as is suggested by the photographs of Krasil'nikov (1963, p. 243). Similarly, in §7, where flow over a wavy wall is compared with Vincenti (1959), non-linearity has an important effect in accentuating the diffusion processes. In § 8, we record how focusing affects modulation in uniform regions, relating energy flux simply to the 'ray tube' law of linear acoustics.

2. Basic equations

In general, processes which are 'slow' compared to a characteristic 'frequency' of the signal are described by treating the gas as a mixture of slowly reacting species. For clarity we shall assume that translational and rotational energy distributions remain everywhere close to equilibrium, and that only one composition variable q, in addition to density $\bar{\rho}$, fluid velocity $\bar{\mathbf{u}}$, and specific internal energy $\bar{e}$, is needed to define the state of the gas. A thermodynamic pressure $\bar{p}$ and specific enthalpy

$$\bar{h} = \bar{e} + \bar{p}/\bar{\rho}$$

may then be defined, and the equations of continuity, momentum and energy written as

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial(\bar{\rho}\bar{u}_j)}{\partial \bar{x}_j} = 0, \tag{1}$$

$$\bar{\rho}\frac{\partial \bar{u}_i}{\partial \bar{t}} + \bar{\rho}\bar{u}_j\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{p}}{\partial \bar{x}_i} = \frac{\partial \bar{\tau}_{ij}}{\partial \bar{x}_j}, \tag{2}$$

$$\bar{\rho}\left(\frac{\partial \bar{h}}{\partial \bar{t}} + \bar{u}_j\frac{\partial \bar{h}}{\partial \bar{x}_j}\right) - \left(\frac{\partial \bar{p}}{\partial \bar{t}} + \bar{u}_j\frac{\partial \bar{p}}{\partial \bar{x}_j}\right) = \bar{\tau}_{ij}\frac{\partial \bar{u}_i}{\partial \bar{x}_j} - \frac{\partial \bar{q}_j}{\partial \bar{x}_j}, \tag{3}$$

where $\bar{p}\delta_{ij} - \bar{\tau}_{ij}$ is the pressure tensor, $\bar{q}_i$ is the energy flux vector, the superposed dashes denote physical variables, and the standard summation convention is used throughout. The rate of composition change at a fluid particle is governed by

$$\bar{\rho}\frac{\partial \bar{q}}{\partial \bar{t}} + \bar{u}_j\frac{\partial \bar{q}}{\partial \bar{x}_j} + \frac{\partial(\bar{\rho}\bar{d}_j)}{\partial \bar{x}_j} = \frac{\bar{\rho}\bar{l}(\bar{p},\bar{\rho},q)}{\tau}, \tag{4}$$

where $\bar{d}_j$ is the diffusion velocity. The vibrational relaxation is towards 'equilibrium states'

$$\bar{l}(\bar{p},\bar{\rho},q) = 0,$$

and has, for all values of $(\bar{p},\bar{\rho},q)$ in the range of interest, a time scale comparable with the 'relaxation time' τ, chosen to make $\bar{l}_q\,(=\partial\bar{l}/\partial q)$ typically of order unity.† For convenience we introduce the specific entropy $\bar{s} = \bar{\eta}(\bar{p},\bar{\rho},q)$ and temperature $\bar{T}$, defined in the standard way so that

$$\bar{T}d\bar{s} = d\bar{h} - d\bar{p}/\bar{\rho}$$

in 'frozen' processes throughout which q remains constant, and introduce non-dimensional variables based on the time scale τ, a representative acoustic speed a_0, density ρ_0 and temperature T_0, such that

$$\bar{t} = \tau t, \quad \bar{x}_i = a_0\tau x_i, \quad \bar{\rho} = \rho_0\rho, \quad \bar{u}_i = a_0 u_i,$$

$$\bar{p} = \rho_0 a_0^2 p, \quad \bar{h} = a_0^2 h, \quad \bar{T} = T_0 T, \quad \bar{s} = T_0^{-1}a_0^2 s,$$

and

$$\bar{l}(\bar{p},\bar{\rho},q) = l(p,\rho,q). \tag{5}$$

† We shall use suffices to denote partial differentiation with respect to state variables and a 'phase variable' α, and dashes to denote ordinary differentiation.

We choose ρ, q and s as independent state variables, defining

$$p = p(h,\rho,q) = \Pi(\rho,q,s), \quad h = H(\rho,q,s),$$
$$l(\Pi(\rho,q,s),\rho,q) = L(\rho,q,s), \quad T = T(\rho,q,s), \quad s = \eta(p,\rho,q),$$

so that the system (1)–(4) may be replaced by

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_j)}{\partial x_j} = 0, \tag{6}$$

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}+\frac{\partial p}{\partial x_i} = \frac{\partial\tau_{ij}}{\partial x_j}, \tag{7}$$

$$T\left(\frac{\partial s}{\partial t}+u_j\frac{\partial s}{\partial x_j}\right)+\Sigma\left(\frac{\partial q}{\partial t}+u_j\frac{\partial q}{\partial x_j}\right) = \rho^{-1}\tau_{ij}\frac{\partial u_i}{\partial x_j}-\rho^{-1}\frac{\partial q_j}{\partial x_j}, \tag{8}$$

$$\frac{\partial q}{\partial t}+u_j\frac{\partial q}{\partial x_j}+\frac{1}{\rho}\frac{\partial(\rho d_j)}{\partial x_j} = L(\rho,q,s), \tag{9}$$

in which Σ is defined by

$$\Sigma(\rho,q,s) = -T\eta_q-(T\eta_p+\rho^{-1})p_q = -\frac{p_q}{p_h}+\frac{\Pi_q}{\Pi_\rho}\frac{p_\rho}{p_h} = \left.\frac{\partial h}{\partial q}\right|_{p,s}, \tag{10}$$

with $$T\eta_p p_h = 1-\rho^{-1}p_h, \quad T\eta_\rho+T\eta_p p_\rho = -\rho^{-1}p_\rho,$$

and $$\Pi_\rho(\rho,q,s) = \frac{p_\rho}{1-\rho^{-1}p_h}. \tag{11}$$

To complete the system (6)–(9) constitutive laws giving τ_{ij}, q_j, d_j in terms of ρ, q, s, u_j and their derivatives must be added. However, the functional form of the relationships has very little effect on the ultrasonic propagation, and it is only for definiteness that we use the expressions

$$\tau_{ij} = \frac{1}{\rho_0 a_0^2}\bar{\tau}_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)+\mu\mu_B\frac{\partial u_k}{\partial x_k}\delta_{ij}, \tag{12}$$

$$q_i = \frac{1}{\rho_0 a_0^3}\bar{q}_i = -\mu K\frac{\partial T}{\partial x_i}+\rho M d_i, \tag{13}$$

$$d_i = \frac{1}{a_0}\bar{d}_i = -\mu D\frac{\partial q}{\partial x_i}, \tag{14}$$

(Hirshfelder, Curtiss & Bird 1964, chapter 11) for a single vibrational relaxation process in which thermal diffusion and its inverse, the Dufour effect, are absent. Here μ^{-1} is the Reynolds number based on a_0, τ and ρ_0, μ_B is the ratio of bulk and shear viscosities, whilst K^{-1}, D^{-1} are Prandtl and Schmidt numbers respectively. There is no restriction on the departure from vibrational equilibrium, and, so long as spatial gradients of ρ, q, s and $\mathbf{u}$ are limited, all diffusion effects are small with μ. Thus, our theory describes propagation through a 'background' flow in which relaxation and diffusion rates are small, except possibly in certain boundary layers.

3. The asymptotic procedure

The analysis is best motivated by analogy with linear acoustics. There, solutions for a vector $\mathbf{w}(\mathbf{x}, t)$ of dependent variables having the form

$$\mathbf{w} = \mathrm{Re}\{\mathbf{W}(\mathbf{x}, t) \exp i(\omega t - \mathbf{k}.\mathbf{x})\}, \quad \mathbf{k}, \omega \,\text{real}, \tag{15}$$

are sought by a procedure which requires that the scales of $\mathbf{x}$ and t should be comparable with those over which the amplitudes $\mathbf{W}$, frequency ω and wavenumber $\mathbf{k}$ vary significantly. Thus ω and $|\mathbf{k}|$ are large, so that the 'phase' $(\omega t - \mathbf{k}.\mathbf{x}) \equiv \alpha$ varies rapidly with $\mathbf{x}$ and t. These essential assumptions

$$\left|\frac{\partial \alpha}{\partial x_i}\right| \gg 1, \quad \frac{\partial \alpha}{\partial t} \gg 1$$

we retain in our non-linear theory, although we discard the irrelevant restriction to wave forms essentially sinusoidal in space and time.

We follow previous investigations of rapidly oscillating signals (Parker 1968) by writing dependent variables as functions of a 'phase variable' $\alpha(\mathbf{x}, t)$ in addition to $\mathbf{x}$ and t. We choose α to record the rapidly fluctuating details of the ultrasonic signal, and to label monotonically those propagating surfaces over which velocity and density are virtually constant, varying only on the scales of $\mathbf{x}$ and t. Although in previous investigations of such 'relatively undistorted waves' (Varley & Cumberbatch, to be published) the independent variables $(\mathbf{x}, t)$ have been replaced by $\mathbf{x}$ and a 'rapid' variable α, with t subsequently related to $\mathbf{x}$ and α, this procedure has pitfalls. Unless judiciously applied, the approximation process leads to a non-uniformly valid representation, meaningful only near the wavefront of a rapid acoustic disturbance (see Parker & Varley 1968). To avoid this, we retain α as an auxiliary variable, formally independent of $\mathbf{x}$ and t, and we use the consequent freedom to ensure greater validity for the representation.

We modify the derivatives in (6)–(9), (12)–(14) by the transformations

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} - \omega \kappa_i \frac{\partial}{\partial \alpha}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \omega \nu \frac{\partial}{\partial \alpha}.$$

Treating ω^{-1} as a small parameter, we then construct solutions of the modified equations defined for all $(\alpha, \mathbf{x}, t)$. The resulting 'extended' solutions contain physically meaningful solutions to (6)–(9), (12)–(14) when α is regarded as a function of $\mathbf{x}$ and t, no longer freely chosen, but implicitly related to $(\mathbf{x}, t)$ through a solution of

$$\frac{\partial \alpha}{\partial x_i} = -\omega \kappa_i(\alpha, \mathbf{x}, t), \quad \frac{\partial \alpha}{\partial t} = \omega \nu(\alpha, \mathbf{x}, t). \tag{16}$$

These equations define the 'phase surfaces' corresponding to the solution, and will always exist provided that the compatibility conditions

$$\left.\begin{aligned} \frac{\partial \kappa_i}{\partial t} + \omega \nu \frac{\partial \kappa_i}{\partial \alpha} &= \frac{\partial \nu}{\partial x_i} + \omega \kappa_i \frac{\partial \nu}{\partial \alpha}, \\ \frac{\partial \kappa_i}{\partial x_j} - \omega \kappa_j \frac{\partial \kappa_i}{\partial \alpha} &= \frac{\partial \kappa_j}{\partial x_i} - \omega \kappa_i \frac{\partial \kappa_j}{\partial \alpha}, \end{aligned}\right\} \tag{17a, b}$$

are satisfied at all values of $(\alpha, \mathbf{x}, t)$.

Before constructing solutions $\rho, \mathbf{u}, q, s, \boldsymbol{\varkappa}, \nu$ to (17) and the extended forms of (6)–(9), (12)–(14), it is useful to discuss the basic solutions which arise in the formal limit $\omega^{-1} \to 0$, $\mu \to 0$, in which both relaxation and diffusion effects vanish. In this case, equations (6)–(9) become homogeneous of first order in α derivatives, and give solutions $\rho = \rho(\alpha)$, $\mathbf{u} = \mathbf{u}(\alpha)$, etc., depending on α alone, which are simple waves (Courant & Friedrichs 1948, Varley 1965). From the general theory of such disturbances it is known that α is implicitly related to $(\mathbf{x}, t)$ in the form

$$\mathbf{x}\,.\,\mathbf{n}(\alpha) - tc(\alpha) + \Phi(\alpha) = 0, \tag{18}$$

showing that planes with unit normal $\mathbf{n}(\alpha)$ and advancing at speed $c(\alpha)$ carry constant values of each dependent variable. There is considerable choice in the functions $\mathbf{n}(\alpha)$, $c(\alpha)$, but corresponding to each choice there are some combinations, known as Riemann invariants, of the dependent variables which are identically uniform throughout the simple wave. It is easily seen that for equations (6)–(9) the composition q and entropy $s = \eta(p, \rho, q)$ are two such Riemann invariants, whilst ρ, $\mathbf{u}$ are related by

$$\left.\begin{aligned} (\nu - u_j\kappa_j)\rho'(\alpha) - \rho\kappa_j u_j'(\alpha) &= 0,\\ \rho(\nu - u_j\kappa_j)u_i'(\alpha) - \Pi_\rho \kappa_i \rho'(\alpha) &= 0. \end{aligned}\right\} \tag{19}$$

The compatibility condition, or characteristic equation,

$$(\nu - u_j\kappa_j)^2 = \Pi_\rho \kappa_j \kappa_j \tag{20}$$

for (19) shows that for arbitrary $\mathbf{n}(\alpha)$ the wave velocity $c(\alpha)\mathbf{n}(\alpha) \equiv (\kappa_j\kappa_j)^{-1}\nu\boldsymbol{\varkappa}$ of each plane wavelet has intrinsic speed (relative to the gas) of magnitude $(\Pi_\rho)^{\frac{1}{2}}$, independent of the wave normal.

These waves, with parameter α chosen to label the wavelets monotonically, model the behaviour of high frequency signals (ω^{-1} small) in each vicinity.

Amongst the solutions to (19), (20), or the equivalent equation

$$\mathbf{u}'(\alpha) = \pm\frac{(\Pi_\rho)^{\frac{1}{2}}}{\rho}\frac{\boldsymbol{\varkappa}(\alpha)}{|\boldsymbol{\varkappa}(\alpha)|}\rho'(\alpha) = \pm\frac{(\Pi_\rho)^{\frac{1}{2}}}{\rho}\mathbf{n}(\alpha)\rho'(\alpha),$$

the standard unidirectional unsteady ($\mathbf{n}'(\alpha) = 0$) and two-dimensional steady ($\nu = 0$, $\kappa_3 = 0$) solutions are just two of many. However, in providing approximate relationships between fluctuations in $\rho, \mathbf{u}, q, s, \nu$ and $\boldsymbol{\varkappa}$ in each region of a high frequency (ω^{-1} small) signal with restricted diffusive forces ($\omega\mu$ small), the unidirectional waves are sufficient. In such signals, the shock formation lengths and times are much larger than one 'wavelength', when the unit vector $\mathbf{n}(\alpha)$ and density $\rho(\alpha)$ have only small fluctuations about their local mean values $\mathbf{N}, P$. The predominant rapid fluctuations in $\mathbf{u}$ are then in the direction $\mathbf{N}$ and are given by

$$\mathbf{u}(\alpha) \sim \mathbf{U} + \frac{(\Pi_\rho)^{\frac{1}{2}}}{\rho}\mathbf{N}[\rho(\alpha) - P]. \tag{21}$$

This relationship allows us to calculate relaxation and diffusion terms which slowly distort and attenuate the signal, whilst the fluctuations in $c(\alpha)$, a nonlinear effect, are calculated from (20) in terms of $\rho(\alpha)$ and $\mathbf{n}(\alpha)$.

In devising the perturbation procedure for small ω^{-1}, small $\omega\mu$, and rapid fluctuations of restricted amplitude δ, we regard each dependent variable in (6)–(9), (12)–(14) as composed of two parts: one corresponding to the background 'quiet' flow and dependent only on $(\mathbf{x}, t)$, whilst the other is a function strictly periodic in α and with zero mean value at each $(\mathbf{x}, t)$. Thus, we set

$$\rho = P(\mathbf{x},t) + \delta\hat{\rho}(\alpha,\mathbf{x},t), \quad q = Q(\mathbf{x},t) + \hat{q}(\alpha,\mathbf{x},t),$$

$$s = S(\mathbf{x},t) + \hat{s}(\alpha,\mathbf{x},t), \quad \mathbf{u} = \mathbf{U}(\mathbf{x},t) + \hat{\mathbf{u}}(\alpha,\mathbf{x},t),$$

$$\boldsymbol{\kappa} = \mathbf{n}(\alpha,\mathbf{x},t)\,\kappa(\alpha,\mathbf{x},t),$$

where $$\mathbf{n}.\mathbf{n} = 1, \quad \mathbf{n}(\alpha,\mathbf{x},t) = \mathbf{N}(\mathbf{x},t) + \hat{\mathbf{n}}(\alpha,\mathbf{x},t),$$

$$\kappa^{-1} = k^{-1}(\mathbf{x},t)[1+\hat{k}(\alpha,\mathbf{x},t)], \quad \nu/\kappa = c(\alpha,\mathbf{x},t) = C(\mathbf{x},t) + \hat{c}(\alpha,\mathbf{x},t), \tag{22}$$

in equations (6)–(9), (12)–(14) extended to all $(\alpha, \mathbf{x}, t)$, and in (17). Here all fluctuations $\delta\hat{\rho}, \hat{\mathbf{u}}, \hat{q}, \hat{s}$ will be regarded as small perturbations about the quiet flow, so that our first iteration theory differs from classical acoustics only in that the wave velocity $\mathbf{n}c$ is allowed to fluctuate. This, however, has major consequences since fluctuations in $\hat{k}$ need not remain small, and shocks may form.

In the system of governing equations extended to all $(\alpha, \mathbf{x}, t)$, (6)–(9) are modified by the addition of terms

$$\omega[(\nu - \kappa u_j n_j)\delta\hat{\rho}_\alpha - \kappa\rho n_j \hat{u}_{j\alpha}], \tag{23}$$

$$\omega\left[(\nu - \kappa u_j n_j)\rho\hat{u}_{i\alpha} - \kappa\Pi_\rho n_i \delta\hat{\rho}_\alpha - \kappa\Pi_q n_i \hat{q}_\alpha - \kappa\Pi_s n_i \hat{s}_\alpha + \kappa n_j \frac{\partial\tau_{ij}}{\partial\alpha}\right], \tag{24}$$

$$\omega\left[(\nu - \kappa u_j n_j)(T\hat{s}_\alpha + \Sigma\hat{q}_\alpha) + \kappa n_j \frac{\tau_{ij}}{\rho}\hat{u}_{i\alpha} - \frac{\kappa}{\rho} n_j \hat{q}_{j\alpha}\right], \tag{25}$$

$$\omega\left[(\nu - \kappa u_j n_j)\,\hat{q}_\alpha - \kappa n_j d_{j\alpha} - \frac{\kappa}{\rho} n_j d_j \delta\hat{\rho}_\alpha\right], \tag{26}$$

respectively on the left-hand sides. Although, formally, these terms are large $O(\omega)$, their contributions to (6)–(9) will remain bounded if the characteristic equation for $\nu, \boldsymbol{\kappa}$ of (16) is satisfied outside shocks at least to $O(\omega^{-1}, \omega\mu)$, and if as in (21)

$$\hat{\mathbf{u}} = \frac{[\Pi_\rho(P,S,Q)]^{\frac{1}{2}}}{P}\mathbf{N}\delta\hat{\rho} + \hat{\mathbf{v}}(\alpha,\mathbf{x},t), \tag{27}$$

with $\hat{\mathbf{v}}, \hat{\mathbf{n}}, \hat{q}, \hat{s}$ remaining small. For definiteness we choose

$$\nu - \kappa u_j n_j = \kappa(\Pi_\rho)^{\frac{1}{2}}, \quad \text{so that} \quad c = u_j n_j + (\Pi_\rho)^{\frac{1}{2}}, \tag{28}$$

and so that physically meaningful α surfaces are characteristic surfaces for the diffusionless system (6)–(9) (the limit $\omega\mu = 0$). Expressions (22) may be determined iteratively following previous methods (Parker 1968), and to first order in the formal small parameters $\omega^{-1}, \omega\mu, \delta$, give a non-linear acoustics. In what follows, all functions Π, T, Σ, L and their partial derivatives will be evaluated as functions of the 'background values' P, Q, S.

At the first iteration to the extended equations (6)–(9) we insist that the contributions (23)–(26), even if finite, must have negligible mean values. Thus, by averaging the equations over finite ranges of α when terms non-linear in $\hat{\rho}$, $\hat{\mathbf{v}}$, $\hat{q}$, $\hat{s}$, $\hat{\mathbf{n}}$ and $\hat{c}$ are neglected, the basic equations for the background flow are found as

$$\frac{\partial P}{\partial t}+U_j\frac{\partial P}{\partial x_j}+P\frac{\partial U_j}{\partial x_j}=0,$$

$$P\frac{\partial U_i}{\partial t}+PU_j\frac{\partial U_i}{\partial x_j}+\frac{\partial \Pi}{\partial x_i}(P,Q,S)=\frac{\partial T_{ij}}{\partial x_j},$$

$$T(P,Q,S)\left(\frac{\partial S}{\partial t}+U_j\frac{\partial S}{\partial x_j}\right)+\Sigma(P,Q,S)\left(\frac{\partial Q}{\partial t}+U_j\frac{\partial Q}{\partial x_j}\right)=P^{-1}T_{ij}\frac{\partial U_i}{\partial x_j}+P^{-1}\frac{\partial Q_j}{\partial x_j},$$

$$\frac{\partial Q}{\partial t}+U_j\frac{\partial Q}{\partial x_j}+\frac{\partial D_j}{\partial x_j}+P^{-1}D_j\frac{\partial P}{\partial x_j}=L(P,Q,S). \tag{29}$$

These, not unnaturally, show that to this approximation the flow may be any standard relaxing, compressible gas flow satisfying suitable boundary conditions, and with viscous, diffusive and heat conduction effects

$$\left.\begin{aligned} T_{ij}&=\mu\left[\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}-(\tfrac{2}{3}+\mu_B)\frac{\partial U_k}{\partial x_k}\delta_{ij}\right],\\ D_i&=-\mu D\frac{\partial Q}{\partial x_i},\\ Q_i&=-\mu\left[\kappa\frac{\partial T}{\partial x_i}+PMD\frac{\partial Q}{\partial x_i}\right],\end{aligned}\right\} \tag{30}$$

(corresponding to expressions (12)–(14)) which will be small with μ except in boundary layers, shear layers and shocks. Only at subsequent iterations does the acoustic signal affect the background flow, introducing acoustic wind and heating.

4. The acoustic signal

In the acoustic signal $\hat{\rho}$ is the predominant fluctuating variable, with $\hat{\mathbf{u}}$ related in such a way that fluctuations $\hat{\mathbf{v}}$, $\hat{q}$, $\hat{s}$ are negligible at the first iteration. Only the oscillatory contributions

$$\left.\begin{aligned} \hat{\tau}_{ij}&=-2\omega\mu\delta\kappa\hat{\rho}_\alpha[N_iN_j-\delta_{ij}(\tfrac{1}{3}-\tfrac{1}{2}\mu_B)]\frac{(\Pi_\rho)^{\frac{1}{2}}}{P},\\ \hat{q}_i&=\omega\mu\delta\kappa\hat{\rho}_\alpha N_iKT_\rho,\\ \hat{d}_i&=0,\end{aligned}\right\} \tag{31}$$

to the diffusive 'forces' (using (12)–(14), with temperature variations in the coefficients μ, μ_B, D, K neglected for simplicity) have a significant effect.

To determine a propagation equation for the basic signal $\hat{\rho}(\alpha,\mathbf{x},t)$ we note that a certain linear combination of (23)–(26) possesses only diffusive terms. We take the similar combination

$$(6)+\frac{n_i}{(\Pi_\rho)^{\frac{1}{2}}}(7)+\frac{\Pi_s}{T\Pi_\rho}(8)+\left(\frac{\Pi_q}{\Pi_\rho}+\frac{\Sigma}{T}\frac{\Pi_s}{\Pi_\rho}\right)(9)$$

of the modified equations (6)–(9) to obtain the exact equation (in this paragraph Π, Σ, T, L and their derivatives are evaluated at (ρ, q, s))

$$\frac{\partial\rho}{\partial t}+(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial\rho}{\partial x_j}+\frac{\rho}{(\Pi_\rho)^{\frac{1}{2}}}n_i\left(\frac{\partial u_i}{\partial t}+(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial u_i}{\partial x_j}\right)+\rho\left(\frac{\partial u_j}{\partial x_j}-n_in_j\frac{\partial u_i}{\partial x_j}\right)$$
$$+\frac{\Pi_q}{\Pi_\rho}\left(\frac{\partial q}{\partial t}+(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial q}{\partial x_j}\right)+\frac{\Pi_s}{\Pi_\rho}\left(\frac{\partial s}{\partial t}+(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial s}{\partial x_j}\right)$$
$$=\left(\frac{\Pi_q}{\Pi_\rho}-\frac{\Sigma}{T}\frac{\Pi_s}{\Pi_\rho}\right)\left(L(\rho,q,s)-\frac{\partial d_j}{\partial x_j}-\frac{d_j}{\rho}\frac{\partial\rho}{\partial x_j}\right)+\frac{\omega\kappa}{\rho}\left(\frac{\Pi_q}{\Pi_\rho}-\frac{\Sigma}{T}\frac{\Pi_s}{\Pi_\rho}\right)n_j(\rho d_j)_\alpha$$
$$+\frac{n_i}{(\Pi_\rho)^{\frac{1}{2}}}\frac{\partial\tau_{ij}}{\partial x_j}+\frac{\Pi_s}{T\Pi_\rho}\left(\tau_{ij}\frac{\partial u_i}{\partial x_j}-\frac{\partial q_j}{\partial x_j}\right)-\frac{\omega\kappa}{\rho}\left\{\rho\frac{n_in_j}{(\Pi_\rho)^{\frac{1}{2}}}\frac{\partial\tau_{ij}}{\partial\alpha}+\frac{\Pi_s}{T\Pi_\rho}n_j(\tau_{ij}\hat{u}_{i\alpha}-\hat{q}_{j\alpha})\right\}. \tag{32}$$

This may be regarded as an ordinary differential equation for ρ along the space-time trajectories

$$d\mathbf{x}:dt=\mathbf{u}+(\Pi_\rho)^{\frac{1}{2}}\mathbf{n}:1 \tag{33}$$

at each α, and is recognizable, in the diffusionless limit $\omega^2\mu = 0$ in which the system (6)–(9) becomes totally hyperbolic, as the bicharacteristic equation along trajectories (33) lying within the characteristic hypersurfaces

$$\left(\frac{d\mathbf{x}}{dt}-\mathbf{u}\right)^2=\Pi_\rho. \tag{34}$$

A similar equation for the ray vector $\mathbf{n}$ is thus to be expected. Since (17*b*), which is a consequence of (17*a*) and suitable initial conditions, implies with (28) that

$$(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial(\kappa n_i)}{\partial x_i}=(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial(\kappa n_j)}{\partial x_i}+\omega\kappa^2\left(c\frac{\partial n_i}{\partial\alpha}-n_i(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial n_j}{\partial\alpha}\right),$$

whilst (17*a*) becomes

$$\frac{\partial(\kappa n_i)}{\partial t}+\frac{\partial(\kappa c)}{\partial x_i}+\omega\kappa^2\left(c\frac{\partial n_i}{\partial\alpha}-n_i\frac{\partial c}{\partial\alpha}\right)=0,$$

suitable equations are readily derived. Indeed, we find that

$$\frac{\partial(\kappa n_i)}{\partial t}+(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial(\kappa n_i)}{\partial x_j}=-\kappa n_j\frac{\partial u_j}{\partial x_i}-\kappa\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial x_i}+\omega\kappa^2n_i\left(n_j\frac{\partial u_j}{\partial\alpha}+\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial\alpha}\right), \tag{35}$$

$$\frac{\partial(\kappa c)}{\partial t}+(u_j+(\Pi_\rho)^{\frac{1}{2}}n_j)\frac{\partial(\kappa c)}{\partial x_j}=\kappa n_j\frac{\partial u_j}{\partial t}+\kappa\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial t}+\omega\kappa^2c\left(n_j\frac{\partial u_j}{\partial\alpha}+\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial\alpha}\right). \tag{36}$$

These also involve directional derivatives along (33), being analogous to the 'strip conditions' satisfied, along each bicharacteristic curve, by the normals $(\phi_i, \phi_t) = (\partial\phi/\partial x_i, \partial\phi/\partial t)$ to characteristic hypersurfaces $\phi = 0$ (34) for which

$$(\phi_t+u_j\phi_j)^2=\Pi_\rho\phi_i\phi_i$$

(see Courant & Hilbert 1965).

Upon substituting (22) into (32), (35), approximating, and using the results

(29)–(31), we obtain after some manipulation (in regions where $\mu^{-1}T_{ij}$, $\mu^{-1}Q_i$, $\mu^{-1}D_i$ remain bounded) the 'propagation equations'

$$2\left\{\frac{\partial\hat{\rho}}{\partial t}+(U_j+(\Pi_\rho)^{\frac{1}{2}}N_j)\frac{\partial\hat{\rho}}{\partial x_j}\right\}+\hat{\rho}\left[\frac{P}{(\Pi_\rho)^{\frac{1}{2}}}\right]\left\{\frac{\partial}{\partial t}\left[\frac{(\Pi_\rho)^{\frac{1}{2}}}{P}\right]+(U_j+(\Pi_\rho)^{\frac{1}{2}}N_j)\frac{\partial}{\partial x_j}\left[\frac{(\Pi_\rho)^{\frac{1}{2}}}{P}\right]\right\}$$
$$+\hat{\rho}\frac{\partial}{\partial\rho}\left(\frac{\Pi_q}{\Pi_\rho}\right)\left(\frac{\partial Q}{\partial t}+U_j\frac{\partial Q}{\partial x_j}\right)+\hat{\rho}\frac{\partial}{\partial\rho}\left(\frac{\Pi_s}{\Pi_\rho}\right)\left(\frac{\partial S}{\partial t}+U_j\frac{\partial S}{\partial x_j}\right)$$
$$+\hat{\rho}\left\{\frac{\partial U_j}{\partial x_j}+N_j\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial x_j}+(\Pi_\rho)^{\frac{1}{2}}\frac{\partial N_j}{\partial x_j}\right\}-\hat{\rho}\frac{\partial}{\partial\rho}\left\{\left(\frac{\Pi_q}{\Pi_\rho}-\frac{\Sigma}{T}\frac{\Pi_s}{\Pi_\rho}\right)L\right\}$$
$$+N_iN_j\frac{\partial U_i}{\partial x_j}+N_j\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial x_j}$$
$$=\omega^2\mu\frac{\kappa}{P}(\kappa\hat{\rho}_\alpha)_\alpha(\tfrac{4}{3}+\mu_B)+\omega^2\mu\frac{\kappa}{P}(\kappa\hat{\rho}_\alpha)_\alpha K\frac{\Pi_s}{\Pi_\rho}\frac{T_\rho}{T}+O(\omega^{-1},\omega\mu,\omega^2\mu\delta) \quad (37)$$

for $\hat{\rho}$, and

$$(1+\hat{\kappa})\left\{\frac{\partial(kN_i)}{\partial t}+(U_j+(\Pi_\rho)^{\frac{1}{2}}N_j)\frac{\partial(kN_i)}{\partial x_j}\right\}-kN_i\left\{\frac{\partial\hat{\kappa}}{\partial t}+(U_j+(\Pi_\rho)^{\frac{1}{2}}N_j)\frac{\partial\hat{\kappa}}{\partial x_j}\right\}$$
$$=-(1+\hat{\kappa})\,k\left\{N_j\frac{\partial U_j}{\partial x_i}-\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial x_i}\right\}+\omega\delta k^2N_i\left(\frac{(\Pi_\rho)^{\frac{1}{2}}}{P}+\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial\rho}\right)\hat{\rho}_\alpha+O(\omega^{-1},\omega\mu,\omega^2\mu\delta) \quad (38)$$

for $k, N_i, \hat{\kappa}$. Here, when the diffusion parameter $\omega^2\mu$ remains bounded, the 'propagation rays' along which (37) will be integrated for $\hat{\rho}$ depend, at each α, only on the background flow variables and the vector $N_i(\mathbf{x}, t)$. They are determined by averaging (38) with respect to α to obtain

$$\frac{\partial(kN_i)}{\partial t}+(U_j+(\Pi_\rho)^{\frac{1}{2}}N_j)\frac{\partial(kN_i)}{\partial x_j}=-kN_j\frac{\partial U_j}{\partial x_i}-k\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial x_i}, \quad (39)$$

and are the rays
$$\frac{dx_i}{dt}=U_i+(\Pi_\rho)^{\frac{1}{2}}N_i \quad (40)$$

along which k, N_i vary according to

$$\frac{d(kN_i)}{dt}=-kN_j\frac{\partial U_j}{\partial x_i}-k\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial x_i}. \quad (41)$$

By comparison of (41), and the similar equation deduced from (36) for kC ($=kU_jN_j+k(\Pi_\rho)^{\frac{1}{2}}$), with the standard parametric system of equations (Courant & Hilbert 1965)

$$\frac{dx_i}{dl}=F_{\phi_i}=2U_i(\phi_t+U_j\phi_j)-2\Pi_\rho\phi_i,$$
$$\frac{dt}{dl}=F_{\phi_t}=2(\phi_t+U_j\phi_j),$$
$$\frac{d\phi_i}{dl}=-F_{x_i}=-2(\phi_t+U_j\phi_j)\,\phi_k\frac{\partial U_k}{\partial x_i}+\phi_j\phi_j\frac{\partial\Pi_\rho}{\partial x_i},$$
$$\frac{d\phi_t}{dl}=-F_t=-2(\phi_t+U_j\phi_j)\,\phi_k\frac{\partial U_k}{\partial t}+\phi_j\phi_j\frac{\partial\Pi_\rho}{\partial t},$$

defining bicharacteristic strips of a system of equations whose characteristic condition is

$$F(x_i, t, \phi_i, \phi_t) \equiv (\phi_t + U_j \phi_j)^2 - \Pi_\rho \phi_j \phi_j = 0,$$

we see that the rays (40), (41) are those of linearized acoustics in the diffusionless case ($\mu \equiv 0$). Here the normals to the characteristic hypersurfaces ϕ = const. have been written in the form

$$(\phi_j, \phi_t) \equiv \left(\frac{\partial \phi}{\partial x_j}, \frac{\partial \phi}{\partial t}\right) = (kN_j, kC).$$

In this theory, neither class (i) nor class (ii) mechanisms affect the propagation rays. Once the background flow (29) is found, only ordinary differential equations (40), (41) need be solved to give the rays, and hence $\mathbf{N}, k$. For convenience we may label these rays as $\mathbf{X}$ = const. and take $(\mathbf{X}, t)$ as independent variables. Then, for a signal whose $\mathbf{N}(\mathbf{x}, 0)$ are normals to the surfaces $\psi(\mathbf{x}, \beta) = 0$ with $\beta \sim$ physical $\omega\alpha$ at $t = 0$, we write $x_i = x_i^*(X_j, t)$, and solve (40), (41)

$$\frac{\partial x_i^*(\mathbf{X}, t)}{\partial t} = U_i + (\Pi_\rho)^{\frac{1}{2}} N_i, \quad \frac{\partial N_i(\mathbf{X}, t)}{\partial t} = (N_i N_j - \delta_{ij}) \left(N_k \frac{\partial U_k}{\partial x_j} + \frac{\partial (\Pi_\rho)^{\frac{1}{2}}}{\partial x_j}\right), \tag{42}$$

for initial conditions

$$\mathbf{x}^*(\mathbf{X}, 0) = \mathbf{X}, \quad kN_i(X_j, 0) = -\frac{\partial \psi}{\partial x_i}.$$

The standard theory of characteristic surfaces then ensures that at all subsequent times $\mathbf{N}(\mathbf{X}, t)$ are normals to propagating wave surfaces

$$\mathbf{x} = \mathbf{x}^*(\mathbf{X}, t) \quad \text{with} \quad \psi(\mathbf{X}, \beta) = 0,$$

whose successive positions compose the characteristic hypersurfaces β = const. formed from the bicharacteristic strips. Each wave surface may then be parametrized like the wave fronts of Varley & Cumberbatch (1965) by t and two functions of $\mathbf{X}$, whilst comparison with (16) shows that for *all* time

$$\beta \sim \text{physical}\, \omega\alpha$$

and is the appropriate large scale phase variable.

5. Modulation, attenuation and distortion

The purely geometric focusing effects and the convection and refraction due to the background flow (29) are embodied in (40), (41), (42) of the previous section. The damping and distortion effects require more detailed analysis, and involve the variable α.

When $(\mathbf{X}, t)$ (or, indeed, X_1, X_2, β, t) are used as independent variables the relevant approximation to (37) takes the form

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\rho} \frac{\partial m}{\partial t}(\mathbf{X}, t) = A(\mathbf{X}, t)\, \omega^2 \mu (\kappa^2 \hat{\rho}_{\alpha\alpha} + \kappa\kappa_\alpha \hat{\rho}_\alpha) = A\omega^2 \mu\kappa (\kappa \hat{\rho}_\alpha)_\alpha \tag{43}$$

and is the growth law for linear acoustics in non-homogeneous, unsteady, relaxing regions, with a diffusive term on the right-hand side. In general (43) is a parabolic equation which is readily transformable into a 'Burgers equation' with variable coefficients, but in the special case $\mu \equiv 0$ when viscous and diffusive damping disappear it reduces to an ordinary differential equation, a 'transport equation'. The behaviour in this limit is simply obtained, and since, when $\omega^2\mu$ is small, we expect diffusion to be negligible almost everywhere, we first construct solutions generalizing Lighthill's (1956) one-dimensional analysis for a non-relaxing gas.

The solutions
$$\hat{\rho}(\alpha, \mathbf{X}, t) = r(\alpha, \mathbf{X}) \exp\{-m(\mathbf{X}, t)\} \tag{44}$$

of the transport equation ((43) with $\mu \equiv 0$) show how $\hat{\rho}$ modulates, along each detailed 'phase ray' $(\mathbf{X}, \alpha) = \text{const}$. Here

$$2\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}\left(\log\left[\frac{(\Pi_\rho)^{\frac{1}{2}}}{P}\right]\right) - \frac{1}{k}\frac{\partial k}{\partial t} + \operatorname{div}(\mathbf{U} + (\Pi_\rho)^{\frac{1}{2}}\mathbf{N}) - \left[\frac{\Pi_q}{\Pi_\rho}L_\rho - \frac{\Pi_s}{\Pi_\rho}\frac{\partial}{\partial \rho}\left(\frac{\Sigma L}{T}\right)\right] \tag{45}$$

(from (29), (37) and (41)) includes terms giving: (*a*) the modulation proportional to $P^{-\frac{1}{2}}(\Pi_\rho)^{\frac{1}{4}}$, due to variations in P, Q, S; (*b*) the modulation with change of 'wavelength', and proportional to $k^{-\frac{1}{2}}$; (*c*) the strengthening and attenuation due to geometrical focusing and non-uniform convection; (*d*) the relaxation damping, with local damping coefficient†

$$\Delta(\mathbf{X}, t) = -\frac{1}{2}\left(\frac{\Pi_q}{\Pi_\rho} - \frac{\Sigma\Pi_s}{T\Pi_\rho}\right)L_\rho + \tfrac{1}{2}L\frac{\Pi_s}{\Pi_\rho}\frac{\partial}{\partial \rho}\left(\frac{\Sigma}{T}\right).$$

Although (45) allows for differences in the sequences of P, U, Q, S values encountered along exact bicharacteristics (34) and their linearizations (40), it does not explicitly exhibit the shock-forming tendencies. Such 'amplitude dispersion' occurs, in our representation, primarily in the function $\hat{\kappa}$ controlling details of the relationship (16) between physical α and $(\mathbf{x}, t)$. However, by subtracting (39) from (38), an appropriate 'transport equation'

$$\frac{\partial \hat{\kappa}(\mathbf{X}, t)}{\partial t} = -\omega\delta k\left(\frac{(\Pi_\rho)^{\frac{1}{2}}}{P} + \frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial \rho}\right)\hat{\rho}_\alpha \tag{46}$$

is obtained for $\hat{\kappa}$. It shows how $\hat{\kappa}$ decreases in compressive regions ($\hat{\rho}_\alpha > 0$), so that κ increases and physical α wavelets converge. Indeed, using (22), (44), (46) we see that
$$\kappa = \frac{k(\mathbf{X}, t)}{1 + \hat{\kappa}(\alpha, \mathbf{X}, t)},$$

where
$$\hat{\kappa} = -\chi(\mathbf{X}, t)\, r_\alpha(\alpha, \mathbf{X}) = -\omega\delta r_\alpha \int_0^t k(\mathbf{X}, t')\frac{e^{-m(\mathbf{X}, t')}}{P}\frac{\partial}{\partial \rho}[\rho(\Pi_\rho)^{\frac{1}{2}}]\,dt'. \tag{47}$$

This gives a familiar form (see, e.g. Varley & Rogers 1967)

$$\max_\alpha\{\omega\delta r_\alpha(\alpha, \mathbf{X})\} < \left\{\int_0^t \frac{ke^{-m}}{P}\frac{\partial}{\partial \rho}\rho(\Pi_\rho)^{\frac{1}{2}}\,dt'\right\}^{-1} = \frac{\omega\delta}{\chi(\mathbf{X}, t)}, \quad \text{all } (\mathbf{X}, t) \tag{48}$$

† When the background flow is at equilibrium $L(P, Q, S) = 0$, equivalent standard expressions (Vincenti & Kruger 1965, p. 257) may be obtained, as in Parker (1968) for one-dimensional flow.

for the restriction on driving signal 'steepness' if shocks are to be prevented, and, for signals of given profile, is a restriction on the product of amplitude *and frequency*.

Equations (44), (47) show the basic structure of the non-linear signal. Along each 'propagation ray' (40) $\hat{\rho}$ varies, with an amplitude modulation factor (A.M.F.) $\exp\{-m(\mathbf{X},t)\}$, which does not depend on the label α for the phase rays. Since, in a portion of physical wave form containing significant fluctuations in $r(\alpha,\mathbf{X})$, β will vary only to $O(\omega^{-1})$, this modulation is the physically observable amplitude modulation. The wave normal varies with $\mathbf{N}(\mathbf{X},t)$, the length scale with $k(\mathbf{X},t)$ (so that 'wavelength' and 'frequency' modulate with k^{-1}, Ck respectively), whilst the detailed distortion of the profile is given *via* the implicit solutions to (16), relating α to $(\mathbf{X},t)$.

To this approximation (16) gives

$$\frac{\partial\alpha}{\partial x_i} = -\frac{\omega k N_i}{1+\hat{\kappa}}, \quad \frac{\partial\alpha}{\partial t} = \frac{\omega k C}{1+\hat{\kappa}}, \tag{49}$$

with $\hat{\kappa}$ given by (47), showing that on some physical wavelets the wave form may steepen without bound. As κ becomes large, the diffusive contributions (31) to (23)–(26) grow (since the true diffusion parameter in (37) and (43) is $\mu\omega^2\kappa^2(\alpha,\mathbf{X},t)$, not $\omega^2\mu$) and the simple wave approximations (27), (28) are invalidated. However, unless we are interested in the details of the shock structure, it is sufficient to say that certain wavelets 'disappear' into shock discontinuities. The solutions will be composed, like Lighthill's (1956) asymptotic solutions of the Burgers equation with vanishingly small viscosity, of regions of relaxing signal (44) separated by, and matched across, certain thin propagating shock layers.

The class (ii) damping mechanisms act predominantly 'within' the shocks, yet their effect on the signal decay is virtually independent of the dissipation mechanism. The only important detail is the shock 'position', defined, in the regions where solutions to (49) are multivalued, by the matching between values $\alpha^-(\mathbf{X},t)$, $\alpha^+(\mathbf{X},t)$ immediately preceding and following the mathematical discontinuity. These are related through (49) by

$$\int_{\alpha^-}^{\alpha^+}[1+\hat{\kappa}(\alpha,\mathbf{X},t)]\,d\alpha = \alpha^+-\alpha^-+\int_{\alpha^-}^{\alpha^+}\hat{\kappa}(\alpha,\mathbf{X},t)\,d\alpha = 0,$$

or, using (47),
$$\frac{\alpha^+-\alpha^-}{r(\alpha^+,\mathbf{X})-r(\alpha^-,\mathbf{X})} = \chi(\mathbf{X},t). \tag{50}$$

We may also derive, for a 'weak shock' matching flow variables $(\hat{\rho}^-,\hat{u}^-)$ to $(\hat{\rho}^+,\hat{u}^+)$ on wavelets with speeds $C+\hat{c}^-$, $C+\hat{c}^+$, 'jump' conditions giving the standard result (Friedrichs 1948, Lighthill 1960, p. 56) that the shock propagates with the mean speed $C+\frac{1}{2}(\hat{c}^++\hat{c}^-)$. Since this condition ensures that

$$\begin{aligned}(1+\hat{\kappa}^+)\frac{\partial\alpha^+}{\partial t} &= -(1+\hat{\kappa}^-)\frac{\partial\alpha^-}{\partial t} = \omega k\frac{(\hat{c}^+-\hat{c}^-)}{2}\\ &= k\left(\frac{(\Pi_\rho)^{\frac{1}{2}}}{P}+\frac{\partial(\Pi_\rho)^{\frac{1}{2}}}{\partial\rho}\right)\omega\delta\tfrac{1}{2}e^{-m}[r(\alpha^+,\mathbf{X})-r(\alpha^-,\mathbf{X})]\\ &= \tfrac{1}{2}[r(\alpha^+,\mathbf{X})-r(\alpha^-,\mathbf{X})]\frac{\partial\chi}{\partial t},\end{aligned}$$

we have the equation

$$\int_{\alpha^-}^{\alpha^+} r(\alpha, \mathbf{X})\, d\alpha = \chi(\mathbf{X}, t)\, \tfrac{1}{2}[r^2(\alpha^+, \mathbf{X}) - r^2(\alpha^-, \mathbf{X})] = (\alpha^+ - \alpha^-)\, \tfrac{1}{2}[r(\alpha^+, \mathbf{X}) + r(\alpha^-, \mathbf{X})] \tag{51}$$

which, with (48), relates $\alpha^+(\mathbf{X}, t)$ and $\alpha^-(\mathbf{X}, t)$ along each shock.

Equation (51) is independent of relaxation effects. In fact it is 'Whitham's (1952) area rule' involving only the initial physical profile $\hat{\rho}(\alpha, \mathbf{X}, 0) = r(\alpha, \mathbf{X})$ along each phase ray, and for waves of arbitrary form it has two equivalent graphical constructions (see Lighthill 1956). Its versatility in solving problems of equilibrium gas dynamics is well known. Here we show how it gives simple and useful results for ultrasonic attenuation.

After shock formation, the solution (44), (47) must be reinterpreted. From each point $(\alpha, \mathbf{X}, t)$ where (50) has roots $\alpha^- = \alpha^+$, and

$$r_\alpha(\alpha, \mathbf{X})\, \chi(\mathbf{X}, t) = 1,$$

a solution pair $\alpha^-(\mathbf{X}, t)$, $\alpha^+(\mathbf{X}, t)$ to (50), (51) emanates. Thereafter each interval (α^-, α^+) is discarded from the solution (44), (47), so that only the intervening portions of signal survive. In these portions the only effects of relaxation occur to this approximation, in the A.M.F. e^{-m} and distortion factor χ, whilst the pairings between α^- and α^+ along each ray $\mathbf{X} = \text{const.}$ are unchanged. The explicit effects are seen by comparing functions $m(\mathbf{X}, t)$, $\chi(\mathbf{X}, t)$ corresponding to a solution of (29) and (39), and the functions $m_0(\mathbf{X}, t)$, $\chi_0(\mathbf{X}, t)$ corresponding to non-relaxing acoustics with the same initial conditions, and in the same (equilibrium) background flow. From (45) and (47) we have

$$\left.\begin{aligned} m(\mathbf{X}, t) - m_0(\mathbf{X}, t) &= \int_0^t \Delta(\mathbf{X}, t')\, dt', \\ \frac{\partial \chi}{\partial t} &= e^{-(m - m_0)} \frac{\partial \chi_0}{\partial t}, \end{aligned}\right\} \tag{52}$$

giving the relaxation damping, which is unaffected by focusing and which in uniform flows gives an exponential decay along each ray. Although the distortion factor $\chi(t)$ (and also equation (50)) is altered by relaxation, equation (51) is unchanged, so that if $\alpha_0^-(\mathbf{X}, t)$, $\alpha_0^+(\mathbf{X}, t)$ give the shock positions in a non-relaxing flow, the corresponding shocks in *any* relaxing flow must be given by some sequence

$$\alpha_0^-(\mathbf{X}, t_0(t)), \quad \alpha_0^+(\mathbf{X}, t_0(t)), \tag{53}$$

of the same pairings, with t_0 related to t by

$$\chi(\mathbf{X}, t) = \chi_0(\mathbf{X}, t_0(t)). \tag{54}$$

The shock growth along each ray is merely retarded by relaxation, and the same solution given by Whitham's area rule serves for *all* relaxing gases.

Equation (52) shows that class (i) relaxation always reduces the distortion factor, so that

$$t_0(t) < t$$

but more significantly $\chi(t)$ will often approach different limits at large t. In such cases some of the wavelets, which in equilibrium signals disappear into shocks, survive unmolested. In 'strong' (or, alternatively, extremely rapid) signals, virtually all wavelets are eventually swept into the shocks and the signal decays like a series of non-relaxing N waves with class (ii) mechanisms providing the predominant dissipation, whilst for weak waves of moderate frequency relaxation damping is the major attenuation mechanism—acting equally at all wavelets of the signal.

Outside shocks, the relative importance of class (i) and class (ii) damping mechanisms is measured by the parameter $\omega^2\mu$. Relaxation damping and diffusive damping are always appreciable over travel times $O(1)$ and $O(\mu^{-1}\omega^{-2})$ respectively, whilst diffusion is accentuated by non-linearity unless $\omega^{-1}\delta^{-1}$ significantly exceeds both of these. Thus, for waves of restricted amplitude

$$\delta < \omega^{-1} \quad \text{and} \quad \delta < \mu\omega$$

(or, when $\partial[\rho(\Pi_\rho)^{\frac{1}{2}}]/\partial\rho \doteqdot 0$)† non-linear dispersion is negligible, $\hat{\kappa}$ remains small, and the linear theory results are a good approximation. For all frequencies ω, (43) may be treated as the linear equation

$$\frac{1}{\omega^2\mu}\frac{1}{A(\mathbf{X},t)}\frac{\partial}{\partial t}(\hat{\rho}e^m) = \frac{\partial^2(\hat{\rho}e^m)}{\partial\alpha^2} \tag{55}$$

for dispersive signals in a non-homogeneous medium. In particular, for sinusoidal signals $\hat{\rho}(\mathbf{X},0,\alpha) = \hat{\rho}_0(\mathbf{X})\sin 2\pi\alpha$ the solution

$$\hat{\rho}(\mathbf{X},t,\alpha) = \exp-\left\{m(\mathbf{X},t)+\omega^2\mu 4\pi^2\int_0^t A(\mathbf{X},t')\,dt'\right\}\hat{\rho}_0(\mathbf{X})\sin 2\pi\alpha$$

shows the added attenuation due to diffusion, verifying that for extremely rapid signals diffusion is the main damping mechanism. In a similar way, a diffusion damping correction to m is obtained at each α along each phase ray, by substituting (44), (47) into (43). Typically it gives a damping coefficient $O(\omega^2\mu)$.

In many problems information about energy propagation is required. For acoustic signals (22), (27), (31) satisfying (37), (39) with $\hat{\rho}(\alpha,\mathbf{X},t)$ having unit period in α for all $(\mathbf{X},t)$ energy propagates with intensity

$$(\mathbf{U}+(\Pi_\rho)^{\frac{1}{2}}\mathbf{N})\frac{\Pi_\rho}{P}\int_0^1 \delta^2\hat{\rho}^2(\alpha,\mathbf{X},t)\,[1+\hat{\kappa}(\alpha,\mathbf{X},t)]\,d\alpha \tag{56}$$

along the propagation rays, where the integral in (56) is taken only over the surviving portions of $(0,1)$ once shocks have formed. Since, with (44), (47), the integral in (56) becomes

$$\delta^2\exp\{-2m(\mathbf{X},t)\}\int r^2(\alpha,\mathbf{X})\,[1-\chi(\mathbf{X},t)\,r_\alpha(\alpha,\mathbf{X})]\,d\alpha,$$

we see that attenuation is strictly proportional to e^{-2m} until shocks form, and thereafter is considerably greater as diffusion mechanisms become significant. The differing damping effects thus become manifest at different stages of signal propagation, the class (i) mechanisms giving damping of magnitude Δ everywhere, whilst (for $\mu\omega^2 \ll 1$) the class (ii) damping is delayed for times $O(\omega^{-1}(\delta\hat{\rho}_{\max})^{-1})$ and then has an appreciable effect.

† In this case the gas behaves like the (physically suspect) Chaplygin gas $p = A - B/\rho$.

6. Uni-directional waves

We consider a one-dimensional disturbance produced by a fluctuating pressure

$$p(0,x_2,x_3,t) = \Pi_0 + a\sin 2\pi\omega t$$

at $x_1 = 0$, and propagating into the uniform, equilibrium, static region $x_1 > 0$. This is the situation usually investigated in non-linear acoustics (see, for example, Beyer 1965, Blackstock 1965).

Equations (29) for the background flow are satisfied trivially, with

$$\mathbf{U} = 0,\quad P = P_0,\quad Q = Q_0,\quad S = S_0 \quad\text{and}\quad L(P_0,Q_0,S_0) = 0.$$

To label the phase surfaces, we choose initial conditions

$$\nu(0,x_2,x_3,t) = 1,\quad \kappa(0,x_2,x_3,t) = [\Pi_\rho(P_0,Q_0,S_0)]^{-\frac{1}{2}} = (\Pi_\rho)^{-\frac{1}{2}},\quad \delta = a(\Pi_\rho)^{-1} \qquad (57)$$

compatible with (28), and such that $\alpha = \omega t$ at $x_1 = 0$. The driving conditions then become

$$\hat{\rho}(\alpha,0,x_2,x_3,t) = \sin 2\pi\alpha,\quad \hat{\kappa}(\alpha,0,x_2,x_3,t) = 0. \qquad (58)$$

Equations (42) have simple solutions showing that the wave normals $\mathbf{N} = (1,0,0)$ and wave-numbers $k = (\Pi_\rho)^{-\frac{1}{2}}$ are uniform, and that the propagation rays

$$X_1 = x_1 - (\Pi_\rho)^{\frac{1}{2}}t,\quad X_2 = x_2,\quad X_3 = x_3$$

are rectilinear. Equations (43), (46) then become

$$\frac{\partial\hat{\rho}}{\partial t}(\mathbf{X},t) + \Delta\hat{\rho} = A\omega^2\mu\kappa(\kappa\hat{\rho}_\alpha)_\alpha, \qquad (59)$$

$$\frac{\partial\hat{\kappa}}{\partial t}(\mathbf{X},t) = -\omega\delta B\hat{\rho}_\alpha, \qquad (60)$$

where Δ is constant, and

$$A = \frac{1}{2P}\left\{\tfrac{4}{3} + \mu_B + K\frac{\Pi_s}{\Pi_\rho}\frac{T_\rho}{T}\right\},\quad B = \frac{1}{P(\Pi_\rho)^{\frac{1}{2}}}\frac{\partial}{\partial\rho}[P(\Pi_\rho)^{\frac{1}{2}}].$$

Their solutions (44), (47) for driving conditions (58), are readily found to involve only α and x_1. For convenience, we write them as

$$\hat{\rho} = e^{-\Gamma y_1}\sin 2\pi\alpha,\quad \hat{\kappa} = -\Gamma^{-1}(1-e^{-\Gamma y_1})\cos 2\pi\alpha, \qquad (61)$$

in terms of a new length scale

$$y_1 = 2\pi\omega\delta B(\Pi_\rho)^{-\frac{1}{2}}x_1 = \Gamma^{-1}\Delta(\Pi_\rho)^{-\frac{1}{2}}x_1, \qquad (62)$$

and introducing a new relaxation coefficient

$$\Gamma = (2\pi\omega\delta B)^{-1}\Delta$$

which is the ratio of relaxation effects to non-linear effects.

Using (61) in (16) we determine the implicit relation

$$-\omega(x_1 - (\Pi_\rho)^{\frac{1}{2}}t) = \alpha - \chi(y_1,\Gamma)\sin 2\pi\alpha \qquad (63)$$

giving details of the profile distortion, where

$$\chi(y_1,\Gamma) = (2\pi\Gamma)^{-1}(1-e^{-\Gamma y_1})$$

is the distortion factor. This, in the equilibrium limit ($\Gamma \to 0$) in which non-linearity swamps relaxation, takes the form

$$\chi(y_1, 0) = \chi_0(y_1) = (2\pi)^{-1} y_1.$$

We see that for all ratios Γ the profile distortions are similar, and are periodic in t at each fixed $\mathbf{x}$. Shocks do not form for all values of Γ, but if

$$\Gamma < 1$$

the equation $$2\pi\chi(y_1, \Gamma) = 1$$

has the solution $$y_1 = -\Gamma^{-1} \log(1-\Gamma),$$

for the shock formation distance. This always exceeds the value unity corresponding to the equilibrium limit, and corresponds to travel times

$$\tau^* = -\Delta^{-1} \log(1-\Gamma), \quad t_0(\tau^*) = (2\pi\omega\delta B)^{-1}.$$

After formation the shocks propagate (satisfying (51)) with $\alpha^+ + \alpha^- =$ constant.

Since, in these periodic signals, all shocks are similar, we need discuss only the shock originating at $x_1 = (\Pi_\rho)^{\frac{1}{2}}\tau^*$, $\tau = \tau^*$, and propagating with $\alpha^- = -\alpha^+$. As y_1 increases this absorbs wavelets α^+ and α^- according to

$$\frac{\alpha^+}{\sin 2\pi\alpha^+} = \frac{\alpha^-}{\sin 2\pi\alpha^-} = \chi(y_1, \Gamma), \tag{64}$$

whilst the signal has A.M.F. $e^{-\Gamma y_1}$ and distortion factor $\chi(y_1, \Gamma)$ which also, for each y_1, depend only on Γ. As Γ increases, the exponential decay and retarded non-linear convection become more noticeable, and the shock takes its maximum strength at the moment it absorbs the wavelet $\alpha^+ = (2\pi)^{-1} \cos^{-1} \Gamma$. The number α_Γ of wavelets eventually absorbed by the shock is given by

$$2\pi\alpha_\Gamma = \Gamma^{-1} \sin 2\pi\alpha_\Gamma,$$

and decreases with Γ. On figure 1 the successive wave profiles of one wavelength for all initial amplitudes δ and frequencies ω are illustrated for a gas of arbitrary relaxation coefficient Δ, when $\omega^2\mu \ll 1$. To this first approximation in the frequency parameter ω^{-1} the relaxation time is not comparable with the signal period and all profiles are symmetric. The shapes are the same as for a non-relaxing gas, but occur at positions y_1 depending on Γ, with associated damping $e^{-\Gamma y_1}$.

The energy intensity at $(\mathbf{x}, t)$ is obtained from (56), (64) as

$$E = \delta^2 \frac{\Pi_\rho}{2P} e^{-2\Gamma y_1} \left\{ 1 - 2\alpha^+ + \frac{\sin 4\pi\alpha^+}{2\pi} + \frac{\alpha^+}{3} \sin^2 2\pi\alpha^+ \right\}, \tag{65}$$

where, for $x_1 \leqslant (\Pi_\rho)^{\frac{1}{2}}\tau^*$, α^+ is zero and the final bracket is unity. In this régime relaxation damping is the only effective dissipation mechanism, since wave form distortion has not yet convected wavelets into shocks. At the point where shocks form, the relative intensity is $(1-\Gamma)^2$, corresponding to the points 3 in figures 2, 3. Beyond this point, wavelets are absorbed into the shock, and the bracketed term gives the diffusive dissipation reducing the intensity from the upper to the lower curves in figures 2, 3.

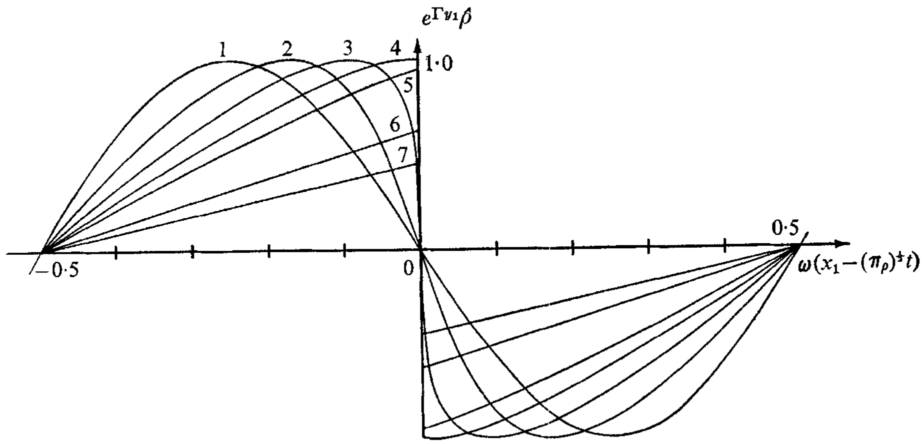

FIGURE 1. Successive profiles of the density signal. $e^{m(\mathbf{X},t)}\hat{\rho} = e^{\Gamma y_1}\hat{\rho}$ against $\omega(x_1 - (\Pi_\rho)^{\frac{1}{2}}t)$ at varying values of the distortion factor $\chi(y_1, \Gamma)$. Curve: 1, $2\pi\chi = 0$; 2, $2\pi\chi = 0{\cdot}5$; 3, $2\pi\chi = 1{\cdot}0$; 4, $2\pi\chi = 1{\cdot}5$; 5, $2\pi\chi = 2{\cdot}0$; 6, $2\pi\chi = 4{\cdot}0$; 7, $2\pi\chi = 6{\cdot}0$.

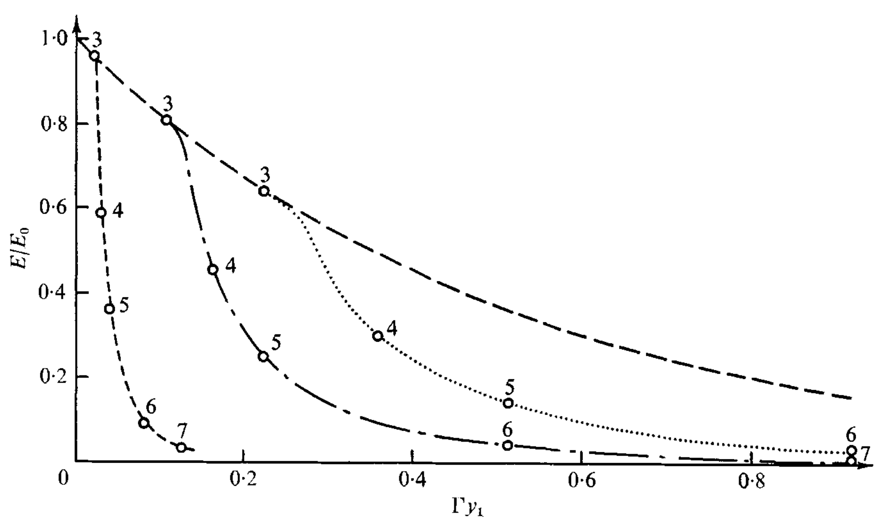

FIGURE 2. Decay of energy intensity with distance for fixed relaxation Δ. E/E_0 plotted against $\Delta(\Pi_\rho)^{-\frac{1}{2}}x_1$, at different frequencies and amplitudes, with the wave form at numbered points corresponding to the profiles in figure 1. — —, —, $\Gamma \geqslant 1$; ......, $\Gamma = 0{\cdot}2$; — — — — —, $\Gamma = 0{\cdot}1$; - - - -, $\Gamma = 0{\cdot}02$.

In figure 2 intensity decay for differing δ, ω in a gas of fixed Δ, B is exhibited. Then $\Gamma^{-1} \sim \omega\delta B$. For weak non-linearity, such that $\Gamma \geqslant 1$, decay is exponential, as though non-linear effects were neglected. For stronger signals shocks form at points 3, and diffusion quickly dominates. The curves for differing Γ are simply related, since, if (64) is expressed as

$$\alpha^+ = \alpha_0^+(2\pi\chi(y_1, \Gamma)),$$

so that for non-relaxing gases

$$\alpha^+ = \alpha_0^+(y_1)$$

and equation (65) becomes

$$\frac{2PE}{\delta^2 \Pi_\rho} = G(\alpha_0^+(y_1)) = g(2\pi\chi_0(y_1)) = g(y_1),$$

the general decay is given by

$$\frac{2PE}{\delta^2 \Pi_\rho} = e^{-2\Gamma y_1} g[\Gamma^{-1}(1-e^{-\Gamma y_1})]. \tag{66}$$

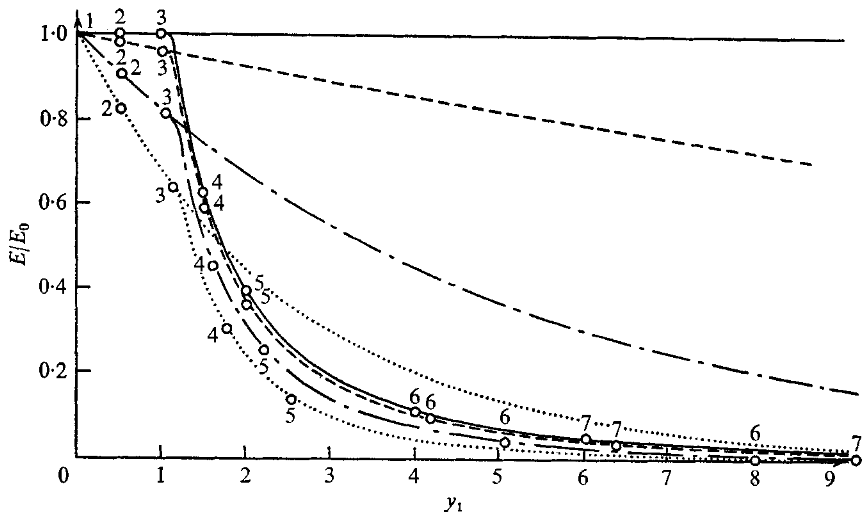

FIGURE 3. Decay of energy intensity with distance for fixed driving amplitude and frequency. E/E_0 plotted against y_1 for varying values of relaxation Γ. The upper curves correspond to linear theory. The lower curves exhibit shock dissipation. ——, $\Gamma = 0$; - - - -, $\Gamma = 0{\cdot}02$; — — — — —, $\Gamma = 0{\cdot}1$; ......, $\Gamma = 0{\cdot}2$.

This information is plotted in a different form in figure 3, where the effect of relaxation for signals of fixed amplitude δ and frequency ω is exhibited. For each Δ (and, hence, each Γ) the upper curve corresponds to linear theory, and the lower curve shows how non-linearity has no effect until shock formation, but has drastic consequences thereafter. The wave forms and intensity decay show good agreement with the photographs of Krasil'nikov (1963) for distortion of ultrasonic waves in water.

7. Flow over a wavy wall

We consider perturbations to the steady supersonic flow $\mathbf{U} = (U_1, U_2, 0)$ at relaxational equilibrium $L(P, Q, S) = 0$, due to a wall W

$$x_3 = \omega^{-1}\delta b(x_1) \sin 2\pi\omega\lambda(x_1) \quad (b > 0,\ \lambda' > 0)$$

having sinusoidal corrugations of slowly varying amplitude and wavelength. Under these conditions the 'propagation rays' (40) are rectilinear and stationary, and may be parametrized by X_1, X_2 as

$$(x_1, x_2, x_3) = \left[X_1 + U_1\left(1-\frac{\Pi_\rho}{U_1^2}\right)l,\ X_2 + U_2 l,\ (\Pi_\rho)^{\frac{1}{2}}\left(1-\frac{\Pi_\rho}{U_1^2}\right)^{\frac{1}{2}} l\right] \tag{67}$$

corresponding to

$$\mathbf{N} = \left[-\frac{(\Pi_\rho)^{\frac{1}{2}}}{U_1}, 0, \left(1-\frac{\Pi_\rho}{U_1^2}\right)^{\frac{1}{2}}\right], \quad k = \frac{U_1}{(\Pi_\rho)^{\frac{1}{2}}}\lambda'(x_1), \tag{68}$$

where l, the propagation time from W to (x_1, x_2, x_3), is independent of the cross flow U_2, and where $U_1(\Pi_\rho)^{-\frac{1}{2}}$, the relevant Mach number, exceeds unity. The appropriate boundary conditions are

$$\delta\hat{\rho} = \lambda'(x_1)\,b(x_1)\,2\pi\delta\frac{U_1 P}{(\Pi_\rho)^{\frac{1}{2}}}\left(1-\frac{\Pi_\rho}{U_1^2}\right)^{-\frac{1}{2}}\cos 2\pi\alpha = \lambda'(x_1)\,b(x_1)\,B_0 r(\alpha), \tag{69}$$

$$\hat{\kappa} = \frac{\hat{u}_1}{U_1} = -\frac{U_1}{(\Pi_\rho)^{\frac{1}{2}}}\left(1-\frac{\Pi_\rho}{U_1^2}\right)^{-\frac{1}{2}}\hat{u}_3 = -\frac{\Pi_\rho}{PU_1^2}\lambda'(x_1)\,b(x_1)B_0 r(\alpha), \tag{70}$$

and in terms of a new length scale

$$Y = 2\pi\omega\delta\frac{U_1^2}{\Pi_\rho}\left(1-\frac{\Pi_\rho}{U_1^2}\right)^{-\frac{1}{2}}\frac{\partial}{\partial\rho}[P(\Pi_\rho)^{\frac{1}{2}}]\,l = \frac{\Delta l}{\Gamma_1}$$

the propagation equations (43), (46) become

$$\frac{\partial\hat{\rho}}{\partial Y} + \Gamma_1\hat{\rho} = \frac{A\Gamma_1}{\Delta}\omega^2\mu\kappa(\kappa\hat{\rho}_\alpha)_\alpha \tag{71}$$

and

$$\frac{\partial\hat{\kappa}}{\partial Y} = -\frac{\Gamma_1}{\Delta}\frac{\omega\delta}{P}\frac{\partial}{\partial\rho}[P(\Pi_\rho)^{\frac{1}{2}}]\,k\hat{\rho}_\alpha$$

or

$$\frac{\partial\kappa^{-1}}{\partial Y} = -\omega\delta\frac{\Gamma_1}{P\Delta}\frac{\partial}{\partial\rho}[P(\Pi_\rho)^{\frac{1}{2}}]\,\hat{\rho}_\alpha, \quad \text{since} \quad 1+\hat{\kappa} = k\kappa^{-1}. \tag{72}$$

Here all coefficients are constant, with Δ the standard relaxation damping coefficient, and A the diffusion coefficient (Lighthill 1956) in acoustic waves.

To compare (71), (72) with Lighthill's treatment for non-relaxing flow we replace α by a new variable $y(\alpha, x_1, Y)$ with

$$y_\alpha = -\omega^{-1}\kappa^{-1},$$

so that

$$k\frac{\partial}{\partial\alpha} \to -\omega^{-1}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial Y} \to \frac{\partial}{\partial Y} + y_Y\frac{\partial}{\partial y}.$$

However, (72) shows that

$$y_Y = \frac{\Gamma_1}{P\Delta}\frac{\partial}{\partial\rho}[P(\Pi_\rho)^{\frac{1}{2}}]\,\delta\hat{\rho} = \frac{\omega^{-1}(\Pi_\rho)^{\frac{1}{2}}}{B_0 U_1}\delta\hat{\rho}$$

and so (71) takes the form

$$\frac{\partial}{\partial Y}(\hat{\rho}e^{\Gamma_1 Y}) + \frac{\Gamma_1}{P\Delta}\frac{\partial}{\partial\rho}[P(\Pi_\rho)^{\frac{1}{2}}]\,\delta\hat{\rho}\frac{\partial}{\partial y}(\hat{\rho}e^{\Gamma_1 Y}) = \mu\frac{A\Gamma_1}{\Delta}\frac{\partial^2}{\partial y^2}(\hat{\rho}e^{\Gamma_1 Y}), \tag{73}$$

like Burgers' equation for $\hat{\rho}e^{\Gamma Y_1}$ with small viscosity and diffusivity, but with exponentially decaying non-linear convection terms. Solutions to (73) with $\Gamma_1 = 0$, Γ_1/Δ finite, and oscillatory data at $Y = 0$ are discussed by Lighthill (1956), verifying that, for small μ, diffusive effects are negligible almost everywhere, and that the signal consists of simple waves matched across various 'shock waves'. In similar solutions for non-zero Γ_1, we expect $\hat{\rho}$ and the 'excess wavelet speed' y_Y to decay with $e^{-\Gamma_1 Y}$ along rays

$$\frac{dy}{dY} = \frac{\Gamma_1}{P\Delta}\frac{\partial}{\partial\rho}[P(\Pi_\rho)^{\frac{1}{2}}]\,\delta\hat{\rho},$$

which are identically the α 'phase rays'. The approximate solution

$$\hat{\rho}(\alpha, x_1, Y) = \lambda'(x_1)\,b(x_1)e^{-\Gamma_1 Y}B_0\cos 2\pi\alpha, \tag{74}$$

matching boundary conditions (69), remains sinusoidal as a function of α, but the excess wavelet speeds of the phase rays will decay. Thus

$$y_\alpha = -\omega^{-1}k^{-1}(1+\hat{\kappa}) \doteqdot \frac{-\omega^{-1}(\Pi_\rho)^{\frac{1}{2}}}{U_1\lambda'(x_1)}\left\{1+\delta b(x_1)\,[\lambda'(x_1)]^2\frac{(1-e^{-\Gamma_1 Y})}{\Gamma_1}2\pi\sin 2\pi\alpha\right\}, \tag{75}$$

and shocks will form only in more violent portions of the signal where

$$2\pi\delta b(x_1)[\lambda'(x_1)]^2 > \Gamma_1.$$

Equation (74) shows the damping effect of class (i) mechanisms everywhere in the signal. Indeed for weak waves with $b(x_1)[\lambda'(x_1)]^2 \ll 1$ in which the phase rays may be taken as linearized bicharacteristics, Vincenti's (1959) linearized results for high frequency waves are recovered. However, for greater amplitudes and frequencies non-linearity may 'feed' portions of the signal into shock regions, where diffusion is accentuated, and the signal is further dampened.

Since (71), (72) are similar to (59), (60) the signal decay (for $\omega^2\mu \ll 1$) is as in figures 1, 2, 3, with Y replacing y_1.

8. Focusing effects

Equation (45) shows separately the contributions to the A.M.F. arising from non-homogeneous background conditions, from refractive and focusing effects, and from relaxation. It generalizes the formula (Bretherton & Garrett 1968) for energy propagation in inhomogeneous moving media to relaxing (but effectively non-dispersive) gases, since $\omega k(\Pi_\rho)^{\frac{1}{2}}$ is their 'intrinsic frequency' in this case.

We illustrate the focusing effects in uniform flow, for which $\mathbf{U} = 0$. Here (39) gives

$$k = k(\mathbf{X}), \quad \mathbf{N} = \mathbf{N}(\mathbf{X}),$$

so that all propagation rays are straight lines

$$\mathbf{x} = \mathbf{X} + (\Pi_\rho)^{\frac{1}{2}}\mathbf{N}(\mathbf{X})t \tag{76}$$

normal to phase surfaces defined at each instant by $\psi(\mathbf{X}, \beta) = 0$, and which may correspond to any family of surfaces at time $t = 0$. Following Varley & Dunwoody (1965) we write

$$\operatorname{div}\mathbf{N} = -2\Omega(\mathbf{X}, t) = \frac{-2\Omega_0(\mathbf{X}) + 2\kappa_0(\mathbf{X})(\Pi_\rho)^{\frac{1}{2}}t}{1-2\Omega_0(\mathbf{X})(\Pi_\rho)^{\frac{1}{2}}t + \kappa_0(\mathbf{X})\Pi_\rho t^2} = \frac{1}{(\Pi_\rho)^{\frac{1}{2}}f}\frac{\partial f}{\partial t},$$

where $\Omega(\mathbf{X},t)$ is the current mean curvature, and $\Omega_0(\mathbf{X})$, $\kappa_0(\mathbf{X})$ are the initial mean and Gaussian curvatures, of the appropriate phase surfaces β = const., and where

$$f = \kappa_0(\mathbf{X})\Pi_\rho t^2 - 2\Omega_0(\mathbf{X})(\Pi_\rho)^{\frac{1}{2}}t + 1$$

records the cross-sectional area of a typical 'ray tube' surrounding $\mathbf{X}$. Then (45) gives

$$e^{-m} = [f(\mathbf{X},t)]^{-\frac{1}{2}}e^{-\Delta t}, \tag{77}$$

showing the intensification $\propto f^{-\frac{1}{2}}$ as rays converge. In (47) this gives

$$\chi \sim \int_0^t [f(X,t')]^{-\frac{1}{2}}e^{-\Delta t'}dt',$$

where the first term arises from focusing. If f increases with t, the corresponding β surfaces are convex, and geometrical attenuation reinforces relaxation, and from (48) the threshold value of r_α for shock formation is raised. However, if $f = 0$ has solutions for positive t, the relevant portion of β surface is concave, and a caustic is reached when $(\Pi_\rho)^{\frac{1}{2}}t$ equals one of the principal curvatures of β = const, $t = 0$. Except for spherically focused waves (in which $f^{\frac{1}{2}}$ is linear in t) χ remains bounded as the caustic is approached, and for sufficiently weak waves the ray theory breaks down before shocks can form.

To this approximation non-linearity cannot delay formation of caustics.

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